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BUTLER GROUPS OF ARBITRARY CARDINALITY

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ABSTRACT

We show that in the constructible universe, the two usual definitions of Butler groups are equivalent for groups of arbitrarily large power. We also prove that $\mathrm{Bext}^2(G, T)$ vanishes for every torsion-free group G and torsion group T. Furthermore, balanced subgroups of completely decomposable groups are Butler groups. These results have been known, under CH, only for groups of cardinalities $\leq \aleph_\omega$.

1. Introduction

All groups in the following are abelian. For standard terminology and notation we refer to the monograph [11].

Butler groups, both in the finite and infinite rank cases, have received much attention in the recent literature: they form a very attractive class of torsion-free groups, abundant with challenging problems. Butler [6] was the first to consider torsion-free groups B of finite rank, now called **Butler groups**, which satisfy either of the following equivalent conditions:

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(a) B is a pure subgroup of a completely decomposable group (of finite rank);

(b) B is an epimorphic image of a completely decomposable group of finite rank.

Groups (b) were also discovered by Bican [3]. Bican–Salce [5] noticed that (a) and (b) were equivalent to the condition that $\mathrm{Bext}^1(B,T) = 0$ for all torsion groups T (where Bext^1 denotes the group of all balanced extensions of T by B). This led to a generalization to torsion-free groups of infinite rank: a torsion-free group B (of any rank) is called a

1. B_1-**group** if $\mathrm{Bext}^1(B,T) = 0$ for all torsion groups T;
2. B_2-**group** if there is a continuous well-ordered ascending chain of pure subgroups,

$$0 = B_0 < B_1 < \cdots < B_\alpha < \cdots < B_\tau = B = \bigcup B_\alpha \tag{1}$$

with rank 1 factors (or, equivalently, with finite rank factors) such that, for each $\alpha < \tau$, $B_{\alpha+1} = B_\alpha + G_\alpha$ holds for some finite rank Butler group G_α.

Bican–Salce [5] proved that these definitions are equivalent for countable groups B and, in general, every B_2-group is a B_1-group. A third class of groups has been introduced by Albrecht–Hill [1]: B is a

3. B_3-**group** if B admits an Axiom-3 family of decent subgroups.

(For definitions see Section 7.) It is an easy exercise to verify that every B_3-group is a B_2-group. Though the claim in [1] that the converse is also true is based on an incorrect proof, the equivalence of B_2- and B_3-groups can be established (see (7.2) infra).

There seems to be a consensus that the major open problems concerning infinite rank Butler groups are as follows: Is every B_1-group a B_2-group (and hence a B_3-group)? Is $\mathrm{Bext}^2(G,T) = 0$ for every torsion-free group G and torsion group T?

For groups of cardinality $\aleph_1$, both questions have affirmative answers; see Dugas–Hill–Rangaswamy [8] and Albrecht–Hill [1], respectively. However, for groups of higher cardinality, the problems are undecidable in ZFC. On one hand, assuming CH, Dugas–Hill–Rangaswamy [8] gave positive answers to the above questions for groups of cardinalities $\leq \aleph_\omega$. On the other hand, Dugas–Thomé [9] have shown that the denial of CH leads to negative answers already at cardinality $\aleph_2$.

Our main objective is to break the barrier $\aleph_\omega$ and to answer the posed questions for groups of arbitrary cardinality at least in the constructible universe L:

THEOREM I: *If $V = L$, then every B_1-group is a B_2-group (and hence a B_3-group).*

THEOREM II: *If $V = L$, then* $\mathrm{Bext}^2(G,T) = 0$ *for all torsion-free groups G and torsion groups T.*

Basically, our approach follows the path opened by the pioneering paper [8] by Dugas–Hill–Rangaswamy. The vehicle for obtaining a chain (1) required for the proof of being a B_2-group was the construction of a chain of separative subgroups with rank one factors. However, this construction failed to work at cardinals which were successors to cardinals cofinal with ω, so they could not pass the cardinal $\aleph_\omega$. One is confronted with a new situation: in abelian group theory, the phenomenon of having $\aleph_\omega$ as a barrier has not occurred before, and therefore it should not be surprising that this question was not susceptible to treatment by any of the set theoretical principles used earlier. The new set theoretical hypothesis which we needed to pass cardinals cofinal with ω is the Box Principle $\square_\lambda$ (known to be a consequence of $V = L$, see (3.1)). Actually, we will be able to establish not only the existence of a separative chain with rank one factors in any torsion-free group (see (4.2)), but also the existence of a richer collection: an Axiom-3 family of separative subgroups; cf. (7.3).

It is noteworthy that the hypothesis $V = L$ can be weakened in Theorems I and II; in fact, all the results can be derived by using only GCH and $\square_\lambda$ (both consequences of $V = L$).

We take advantage of this opportunity to give a simplified, direct approach to the theory of Butler groups of arbitrary cardinality (some proofs in the paper Bican–Fuchs [4] on Bext run parallel to arguments here). The bulk of the proofs reflect a mixture of ideas in the literature, especially in [8], along with our own. In addition, we correct the mistake in [1].

2. Balanced and separative subgroups

As usual, $\chi_G(a)$, or simply $\chi(a)$, will denote the characteristic of an element a in a given group G. A pure subgroup A of the torsion-free group G is said to be a balanced subgroup if every coset $g + A$ ($g \in G$) contains an element $g + a$ ($a \in A$) (called **proper with respect to** A) such that $\chi(g + a) \geq \chi(g + x)$ for each $x \in A$. An exact sequence $0 \to A \to G \to C \to 0$ is balanced-exact if the image of A in G is a balanced subgroup of G. Balanced-exactness of the last exact sequence is equivalent to the property that, for every rank 1 torsion-free group J, every homomorphism $J \to C$ can be lifted to a map $J \to G$.

LEMMA 2.1: *A pure subgroup A of a torsion-free group G is balanced in G if and only if, for every $g \in G$ and for every countable subset $\{a_n\}_{n<\omega}$ of A, there exists an $a \in A$ such that*

$$\chi(a + a_n) \geq \chi(g + a_n) \quad \textit{for every } n < \omega. \tag{2}$$

Proof: Let A be balanced in G and $g \in G$. For some $a \in A$, $g - a$ is proper with respect to A, i.e. $\chi(g - a) \geq \chi(g + x)$ for all $x \in A$. But then $\chi(a + x) \geq \min\{\chi(a - g), \chi(g + x)\} = \chi(g + x)$ for all $x \in A$. Conversely, suppose that for every $g \in G$ and $\{a_n\}_{n<\omega} \subset A$ there is an $a \in A$ satisfying (2). Then $\chi(g - a) \geq \chi(g + a_n)$ for each a_n. If the set $\{a_n\}_{n<\omega}$ is chosen so as to satisfy $\chi_{G/A}(g + A) = \bigcup_n \chi(g + a_n)$, then evidently $\chi_A(g - a) \geq \chi_{G/A}(g + A)$, i.e. $g - a$ is proper with respect to A. ∎

For the next lemma, see [8, Observation 5.3].

LEMMA 2.2: *Let $0 = A_0 < A_1 < \cdots < A_\nu < \cdots$ $(\nu < \lambda)$ be a (not necessarily continuous) well-ordered ascending chain of balanced subgroups of the torsion-free group G. If $\mathrm{cf}\lambda \geq \omega_1$, then $A = \bigcup_\nu A_\nu$ is again balanced in G.*

Proof: Given $g \in G$, there exists a countable subset $\{a_n\}_{n<\omega}$ of A such that $\chi_{G/A}(g + A) = \bigcup_n \chi(g + a_n)$. In view of $\mathrm{cf}\lambda \geq \omega_1$, there is an index $\mu < \lambda$ with $\{a_n\}_{n<\omega} \subset A_\mu$. The balancedness of A_μ implies the existence of an element $a_\mu \in A_\mu$ such that $\chi(g + a_\mu) = \chi_{G/A_\mu}(g + A_\mu) \geq \bigcup_n \chi(g + a_n) = \chi_{G/A}(g + A)$. ∎

We can improve on an important lemma by Dugas–Hill–Rangaswamy [8, Lemma 5.2] that is required later on. The proof is reminiscent of their argument.

LEMMA 2.3: *Let A and $H_m (m < \omega)$ be pure subgroups of the torsion-free group G $(H_0 = 0)$ such that H_m, $A + H_m$ are balanced in G for all $m < \omega$. Given a subgroup C of A, there exists a subgroup B of A such that*

(i) *$C \leq B \leq A$;*

(ii) *$|B| \leq |C|^{\aleph_0}$;*

(iii) *$B + H_m$ is balanced in G for each $m < \omega$.*

Proof: Ignoring a trivial case, assume $\kappa = |C|^{\aleph_0} \geq 2^{\aleph_0}$. Follow the proof in [8] to obtain a balanced subgroup B_0 of A of cardinality $\leq \kappa$ which contains C (it is simpler to use our (2.1) than the argument with hyperbalanced subgroups in [8]). Embed B_0 in a subgroup B_1 with $|B_1| \leq |B_0|^{\aleph_0} \leq \kappa$ such that $(B_1 + H_1)/H_1$ is balanced in G/H_1. Then the balancedness of H_1 in G guarantees that $B_1 + H_1$ will be balanced in G. Continue in this way to obtain an ascending chain of subgroups B_n $(n < \omega)$ of cardinalities $\leq \kappa$ with $(B_n + H_n)/H_n$ balanced in G/H_n. Let B_ω be the union of $\{B_n | \ n < \omega\}$; thus $|B_\omega| \leq \kappa$. Define B_α for $\alpha < \omega_1$ by transfinite recursion as follows. If subgroups B_β of cardinalities $\leq \kappa$ have been defined for all $\beta < \alpha$, and if α is a limit ordinal, then let B_α be the union of the B_β with $\beta < \alpha$. If $\alpha = \gamma + n + 1$ (γ limit ordinal and $n < \omega$), then embed $B_{\gamma+n}$ in a subgroup $B_\alpha = B_{\gamma+n+1}$ of cardinality $\leq \kappa$ such that $(B_\alpha + H_n)/H_n$ is balanced in G/H_n. This $B_\alpha + H_n$ will then be balanced in G. From (2.2) we conclude that the union B of all these B_α $(\alpha < \omega_1)$ satisfies (i)–(iii). ∎

An immediate consequence is:

COROLLARY 2.4: *A torsion-free group that contains no balanced subgroups other than the trivial ones must have cardinality $\leq 2^{\aleph_0}$.* ∎

A pure subgroup A of a torsion-free group G is called **separative** if for each $g \in G$ there is a countable subset $\{a_n | \ n < \omega\} \subset A$ such that $\{\chi(g + a_n) | \ n < \omega\}$ is a cofinal subset in the set $\{\chi(g + a) | \ a \in A\}$. (This concept was introduced by P. Hill under the name 'separable' and used extensively in [1] and [7]–[8].) The following properties are well-known:

LEMMA 2.5:

(a) *If A is a separative subgroup of G and $A \leq B \leq G$ with B/A countable, then B is likewise separative in G.*

(b) *The union of a countable ascending chain of separative subgroups is again separative.* ∎

The following concept has been introduced in [8]. Define a B^0-subgroup of a torsion-free group G to be a balanced subgroup of G. If for some $\alpha < \omega_1$, B^α-subgroups of G have been defined, then define $B^{\alpha+1}$-subgroups to be those subgroups of G which can be obtained as unions of countable ascending chains of B^α-subgroups. If $\alpha < \omega_1$ is a limit ordinal, then B^α-subgroups are all the B^β-subgroups for $\beta < \alpha$. Finally, a B^∞-**subgroup** of G is a subgroup which is a B^α-subgroup for some $\alpha < \omega_1$. We have:

LEMMA 2.6 [8, LEMMA 5.5]:

(a) *The class of B^∞-subgroups is closed under taking unions of countable chains.*

(b) *If C is a B^∞-subgroup of B and B is balanced in G, then C is a B^∞-subgroup of G.*

(c) *B^∞-subgroups of G are separative in G.* ∎

3. The key lemma

As pointed out in the Introduction, the difficulty in establishing the existence of a sufficient supply of separative subgroups in torsion-free groups of cardinality beyond $\aleph_\omega$ lies in passing from a cardinal cofinal with ω to the next cardinal. We focus our attention on this case, as this is the key to answer fully the questions raised in the Introduction for groups of arbitrary cardinalities.

Let λ be a singular cardinal. Jensen [16] proved that the following "Box Principle" holds in L.

$\Box_\lambda$ There exists a family of sets, C_ν, for limit ordinals $\nu < \lambda^+$ such that

(i) C_ν is a cub (i.e. closed and unbounded) in ν;

(ii) the order type of C_ν is $< \lambda$;

(iii) coherence property: if μ is a limit point in C_ν, then $C_\mu = C_\nu \cap \mu$.

If $\mathrm{cf}\,\lambda = \omega$, there is a countable properly ascending chain of regular cardinals $\kappa_n (n < \omega)$ such that $\bigcup \kappa_n = \lambda$. In particular, we have $\lambda^{\aleph_0} = \lambda^+$ and $\kappa_n^{\aleph_0} = \kappa_n$ for all n. (Note that GCH implies that $\kappa^{\aleph_0} = \kappa$ whenever $\mathrm{cf}\kappa > \omega$ and $\kappa^{\aleph_0} = \kappa^+$ if $\mathrm{cf}\kappa = \omega$; cf. Jech [15, p.49].)

Here is the crucial, but rather technical lemma in whose proof we use both GCH and $\Box_\lambda$.

LEMMA 3.1 ($V = L$): *Let G be a torsion-free group, A and H_m $(m < \omega)$ pure subgroups of G such that H_m and $A + H_m$ are balanced in G $(H_0 = 0)$. Suppose that $|A| = \lambda^+$ where $\mathrm{cf}\lambda = \omega$. There are pure subgroups A_α $(\alpha < \lambda^+)$ in A, pure subgroups A_α^n $(n < \omega)$ in each A_α, and in case $\mathrm{cf}\alpha = \omega$, pure subgroups A_α^{nk} $(k < \omega)$ of A_α^n such that*

(a) *for all* $\beta < \alpha < \lambda^+$, $A_\beta \leq A_\alpha$; $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ *if* α *is a limit ordinal;* $A = \bigcup_{\alpha<\lambda^+} A_\alpha$;

(b) *for every* $n < \omega$, $A_\alpha^n \leq A_\alpha^{n+1}$ *and* $A_\alpha = \bigcup_{n<\omega} A_\alpha^n$ *for each* α;

(c) *if* $\mathrm{cf}\alpha = \omega$, $A_\alpha^n = \bigcup_{k<\omega} A_\alpha^{nk}$ *where* $A_\alpha^{nk} \leq A_\alpha^{n\,k+1}$ *for every* $k < \omega$;

(d) $|A_\alpha^n| = \kappa_n$ *(and hence* $|A_\alpha| = \lambda$*) for each* α, n;

(e) *if* $\mathrm{cf}\alpha \neq \omega$, *then* $A_\alpha^n + H_m$ *is balanced in* G *for all* $m, n < \omega$; *if* $\mathrm{cf}\alpha = \omega$, *then for each* $m, n, k < \omega$, $A_\alpha^{nk} + H_m$ *is balanced in* G;

(f) *if* $\alpha = \beta + 1$, *then in case* $\mathrm{cf}\beta \neq \omega$, $A_\alpha^n + A_\beta^k + H_m$ *is balanced in* G *for all* $n, k, m < \omega$; *while in case* $\mathrm{cf}\beta = \omega$, $A_\alpha^n + A_\beta^{kl} + H_m$ *is balanced in* G *for all* $n, k, l, m < \omega$.

Proof: To facilitate induction, we will require that, in addition to (a)–(f), the following conditions be also satisfied by the subgroups to be constructed. $A = \{a_\alpha \mid \alpha < \lambda^+\}$ will denote a well-ordering of the elements of A.

(i) For every $\beta < \alpha < \lambda^+$, $A_\beta^n \leq A_\alpha^n$ if n is large enough;

(ii) if $\alpha = \beta + 1$, then $a_\beta \in A_\alpha$, and for every n, $A_\beta^n \leq A_\alpha^n$;

(iii) if $\alpha = \beta + 1$ and if β is a limit ordinal such that the order type of C_β is $\geq \kappa_n$, then $A_\beta^n = A_\alpha^n$;

(iv) if β is a limit point of C_α, then $A_{\beta+1}^n \leq A_\alpha^n$ for each n;

(v) if β is a limit point of C_α and if the order type of C_β is $\geq \kappa_n$, then $A_\beta^n = A_\alpha^n$;

(vi) if $\mathrm{cf}\alpha = \omega$ and if the order type of C_α is $\geq \kappa_n$, then $A_\alpha^n \leq A_\alpha^{h0}$ for all $h > n$.

The subgroups A_α, A_α^n (as well as A_α^{nk} for $\mathrm{cf}\alpha = \omega$) will be constructed by transfinite induction on α. Four cases are to be distinguished according as $\alpha = 0$, α is a successor ordinal, or α is a limit ordinal such that the set of limit points in C_α is bounded resp. unbounded in α.

CASE 1: $\alpha = 0$. (2.3) allows us to define, by induction on n, subgroups A_0^n such that $A_0^{n-1} \leq A_0^n$, $|A_0^n| = \kappa_n$, and, in addition, the subgroups $A_0^n + H_m$ are balanced in G, for each $n, m < \omega$. Finally, we set $A_0 = \bigcup_{n<\omega} A_0^n$. With this choice, conditions (b), (d) and (e) are satisfied for $\alpha = 0$, the rest are vacuous.

CASE 2: $\alpha = \beta + 1$. If β happens to be a limit ordinal, then let the index t be minimal such that the order type of C_β is $< \kappa_t$; such a t exists in view of $\square_\lambda$(ii). Otherwise let $t = 0$.

For $n < t$, set $A_\alpha^n = A_\beta^n$. Then all of $A_\alpha^n + H_m$ will be balanced in G ($m < \omega$) whenever $\mathrm{cf}\beta \neq \omega$. If $\mathrm{cf}\beta = \omega$, then the order type of C_β must be

$\geq \kappa_n$. Let δ be the κ_nth member of C_β. By induction hypothesis, $A^n_\delta = A^n_\beta$ and $A^n_\delta + H_m$ is balanced for all $m < \omega$, because $\mathrm{cf}\delta = \kappa_n > \omega$.

For $n \geq t$, define A^n_α via induction as follows. Let $\{B_j |\, j < \omega\}$ stand for the set $\{H_m, A^n_\beta + H_m |\, n, m < \omega\}$ or for the set $\{H_m, A^{nl}_\beta + H_m |\, l, n, m < \omega\}$ of balanced subgroups of G according as $\mathrm{cf}\beta \neq \omega$ or $\mathrm{cf}\beta = \omega$. Using (2.3) we can choose A^n_α to be a balanced subgroup of G such that (1) $A^n_\beta, A^{n-1}_\alpha$ and a_β are all contained in A^n_α; (2) $|A^n_\alpha| = \kappa_n$; (3) for all $j < \omega$, the subgroups $A^n_\alpha + B_j$ are balanced in G. Finally, set $A_\alpha = \bigcup_{n<\omega} A^n_\alpha$.

Conditions (a)–(e) are evidently satisfied. (f) requires proof only for $\mathrm{cf}\beta = \omega$, in which case the balancedness of $A^n_\alpha + A^{kl}_\beta + H_m$ is assured by (v): $A^n_\beta \leq A^{h0}_\beta$ for all $h > n$. (i)–(iii) are trivial consequences of our choice, while (iv)–(vi) are vacuously true.

CASE 3: *α is a limit ordinal and the set of limit points in C_α is bounded in α.* In this case $\Box_\lambda$(i) implies that the order type of C_α is of the form $\delta + \omega$ with δ either 0 or a limit ordinal. Hence $\mathrm{cf}\alpha = \omega$. Since C_α is closed in α, either C_α has no limit points at all or has a last limit point, say η (which is clearly the δth limit point in C_α). Choose a sequence $\beta_0 < \beta_1 < \cdots < \beta_i < \cdots$ of successor ordinals with supremum α; without loss of generality, we may put $\beta_0 = \delta + 1$ whenever δ exists.

Define a strictly increasing sequence of non-negative integers n_i as follows. n_0 is the smallest integer t satisfying $\delta < \kappa_t$. If n_{i-1} has been chosen for some $i \geq 1$, then let n_i be the smallest integer $> n_{i-1}$ for which $A^n_{\beta_j} \leq A^n_{\beta_i}$ holds for all $j < i$ and all integers $n \geq n_i$; such an n_i exists because β_i satisfies (i), i.e. $A^n_{\beta_j} \leq A^n_{\beta_i}$ for large enough n. Once the sequence $\{n_i |\, i < \omega\}$ has been defined, we can set

$$A^n_\alpha = A^n_{\beta_i} \quad \text{if } n_i \leq n < n_{i+1},$$

and obviously, $A_\alpha = \bigcup_{n<\omega} A^n_\alpha$. Finally, in case $\mathrm{cf}\alpha = \omega$, let $A^{nk}_\alpha = A^n_\alpha$ for each $k < \omega$.

To verify (a) for α, pick a $\beta < \alpha$. There is an index i such that $\beta \leq \beta_i$. Since $A_\beta \leq A_{\beta_i}$ by induction hypothesis, it suffices to show $A_{\beta_i} \leq A_\alpha$. Let $x \in A_{\beta_i}$, i.e. $x \in A^n_{\beta_i}$ for some n. There is an index j with $n < n_j$ where without loss of generality $i < j$ can be assumed. By the definition of n_j, we have $A^{n_j}_{\beta_i} \leq A^{n_j}_{\beta_j} = A^{n_j}_\alpha \leq A_\alpha$. But $A^n_{\beta_i} \leq A^{n_j}_{\beta_i}$, so $x \in A_\alpha$. From the definition it is clear that $A_\alpha \leq \bigcup_{\beta<\alpha} A_\beta$, thus $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$, indeed.

Turning to (b), for a given n, let i be the minimal index such that $n+1 \leq n_{i+1}$. If here we have strict inequality, then $A_\alpha^n = A_{\beta_i}^n$ and $A_\alpha^{n+1} = A_{\beta_i}^{n+1}$, so we have $A_\alpha^n \leq A_\alpha^{n+1}$. If $n+1 = n_{i+1}$, then by the choice of n_{i+1}, we have $A_{\beta_i}^{n+1} \leq A_{\beta_i+1}^{n+1}$, thus $A_\alpha^n = A_{\beta_i}^n \leq A_{\beta_i}^{n+1} \leq A_{\beta_i+1}^{n+1} = A_\alpha^{n+1}$.

(c) for $\mathrm{cf}\alpha = \omega$ is evident by the choice of the A_α^{nk}, and so is (d) in view of $A_\alpha^n = A_{\beta_i}^n$ for some $i < \omega$.

(e) is satisfied, since β_i is a successor ordinal, and therefore $A_\alpha^n + H_m = A_{\beta_i}^n + H_m$ is a balanced subgroup for each n, $m < \omega$. (f) is meaningless in this case.

Of conditions (i)–(vi), only (i) and (iv)–(vi) are meaningful in Case 3. First consider (i). If $\beta < \alpha$, there is an index i such that $\beta \leq \beta_i$. Since for large enough integers n, $A_\beta^n \leq A_{\beta_i}^n$ holds, it suffices to prove $A_{\beta_i}^n \leq A_\alpha^n$ for large n. But for $n \geq n_i$ we have $A_\alpha^n = A_{\beta_j}^n$ for $n_j \leq n < n_{j+1}$ with $j \geq i$. By the definition of n_j, $n_j \leq n$ implies $A_{\beta_i}^n \leq A_{\beta_j}^n = A_\alpha^n$.

Next concentrate on (iv). Let β be a limit point of C_α. Thus $\beta \leq \eta$, and either $\beta = \eta$ or $\beta \in C_\alpha \cap \eta = C_\eta$ because of $\Box_\lambda$(iii). Recall that by agreement $\beta_0 = \eta + 1$, so for every n we have $A_{\beta_0}^n \leq A_\alpha^n$ (if $n < n_1$, then $A_\alpha^n = A_{\beta_0}^n$, otherwise $n_j \leq n < n_{j+1}$ for some $j \geq 1$, thus $A_\alpha^n = A_{\beta_j}^n$; by the definition of n_j, $A_{\beta_0}^n \leq A_{\beta_j}^n$). Therefore, if $\beta = \eta$, then $A_{\beta+1}^n \leq A_\alpha^n$ for all n. On the other hand, if $\beta < \eta$, then β is a limit point of C_η ($= C_\alpha \cap \eta$), so by the induction hypothesis for η, $A_{\beta+1}^n \leq A_\alpha^n$ for each n. We conclude that $A_{\beta+1}^n \leq A_\eta^n \leq A_{\eta+1}^n \leq A_\alpha^n$ as desired.

In order to check (v), recall that t was chosen to be minimal satisfying $\delta < \kappa_t$ and η is the δth point of C_α. Hence C_η has order type δ. If β is a limit point of C_α and the order type of C_β is $\geq \kappa_n$, then the order type of $C_\alpha \cap \beta$ is $< \kappa_t$, i.e. $n < t$. If $\beta = \eta$, then $A_\alpha^n = A_{\eta+1}^n$ (note that $n < t$ implies $n < n_0 < n_1$). From (iii) for $\eta + 1$, we conclude that $A_{\eta+1}^n = A_\eta^n$ ($\kappa_{t-1} \leq \delta$), thus $A_\alpha^n = A_\eta^n$. If $\beta < \eta$, then β is a limit point of C_η. By (v) for η we have $A_\beta^n = A_\eta^n = A_{\eta+1}^n = A_\alpha^n$.

Finally, to prove (vi) note that A_α^n is balanced and $A_\alpha^n \leq A_\alpha^h$ for $h > n$.

CASE 4: *α is a limit ordinal and the set D_α of limit points in C_α is unbounded in α.* We now define $A_\alpha^n = \bigcup_{\beta \in D_\alpha} A_{\beta+1}^n$ and $A_\alpha = \bigcup_{n<\omega} A_\alpha^n$. If $\mathrm{cf}\alpha = \omega$, then let $\beta_k \in D_\alpha (k < \omega)$ be a cofinal ascending chain in α such that the order type of C_{β_0} is $> \kappa_{t-1}$ (no restriction on the choice of β_0 in case $t = 0$), and set $A_\alpha^{nk} = A_{\beta_k+1}^n$. Observe that if $\gamma < \beta$ in D_α, then by virtue of (v) γ is a limit point of C_β, so $A_{\gamma+1}^n \leq A_\beta^n \leq A_{\beta+1}^n$.

(a) is obvious from $A_\alpha = \bigcup_{\beta \in D_\alpha} A_{\beta+1}$. (b) follows from the inclusion $A_\alpha^n = \bigcup_{\beta \in D_\alpha} A_{\beta+1}^n \le \bigcup_{\beta \in D_\alpha} A_{\beta+1}^{n+1} = A_\alpha^{n+1}$ which is a consequence of $A_{\beta+1}^n \le A_{\beta+1}^{n+1}$ for all $\beta < \alpha$. For $\mathrm{cf}\alpha = \omega$, (c) is satisfied in view of (iv) by the choice of the A_α^{nk}.

To prove (d), again choose t to be minimal such that the order type of C_α is $< \kappa_t$. Then $|D_\alpha| \le \kappa_t$ holds true. Thus for $n \ge t$, $A_\alpha^n = \bigcup_{\beta \in D_\alpha} A_{\beta+1}^n$ is the union of at most κ_t sets, each of cardinality κ_n. Consequently, in this case $|A_\alpha^n| \le \kappa_n$. If $n < t$, then the order type of C_α is $> \kappa_n$ and A_α^n is the union of more than κ_n sets, but we shall show that most of these sets are equal. The κ_nth member β of C_α is clearly a limit point of C_α, so $\beta \in D_\alpha$. The order type of D_β is exactly κ_n. We claim that $A_\alpha^n = A_{\beta+1}^n$ (which is by (iii) equal to A_β^n). Every γ ($\gamma < \beta, \gamma \in D_\alpha$) is a limit point of C_β, so $A_{\gamma+1}^n \le A_\beta^n$ for all n. If $\gamma \in D_\alpha$ and $\gamma > \beta$, then β is a limit point of C_γ, so (v) applied to β, γ implies $A_\gamma^n = A_\beta^n = A_{\beta+1}^n$. The order type of C_γ being $> \kappa_n$, (iii) is applicable, whence $A_\gamma^n = A_{\gamma+1}^n$, and we obtain $A_{\gamma+1}^n = A_{\beta+1}^n$. Thus A_α^n — as the union of equal sets $A_{\gamma+1}^n = A_{\beta+1}^n$ — has cardinality $|A_{\beta+1}^n| = \kappa_n$.

Next we prove (e). For each $n, m < \omega$, $A_{\beta+1}^n + H_m$ is a balanced subgroup in G, so $A_\alpha^n + H_m$ is the union of an ascending chain of balanced subgroups. If $\mathrm{cf}\alpha > \omega$, then $A_\alpha^n + H_m$ is balanced. If $\mathrm{cf}\alpha = \omega$, then the subgroups $A_\alpha^{nk} + H_m = A_{\beta_k+1}^n + H_m$ are balanced in G.

(f), (ii) and (iii) are vacuously satisfied. To see that (i) holds, note that if $\gamma < \alpha$ and if $\beta \in D_\alpha$ is chosen such that $\gamma < \beta$, then for large n we have $A_\gamma^n \le A_{\beta+1}^n \le A_\alpha^n$. (iv) follows from the observation that if β is a limit point of C_α, i.e. if $\beta \in D_\alpha$, then by definition $A_{\beta+1}^n \le A_\alpha^n$. In order to verify (v), argue that if β is a limit point of C_α and the order type of C_β is $\ge \kappa_n$, then the proof of (d) shows that $A_\alpha^n = A_{\beta+1}^n$ which is equal to A_β^n.

Finally, to check (vi), assume $\mathrm{cf}\alpha = \omega$. If the order type of C_α is $\ge \kappa_n$, then necesssarily $n < t$. Let β be the κ_nth member of C_α; then by (v) $A_\alpha^n = A_\beta$. Evidently, $\beta < \beta_0$. Since both β and β_0 are limit points of C_α, we infer that β is a limit point of C_{β_0}, so the inductive hypothesis of (v) for β_0 implies $A_{\beta_0}^n = A_\beta^n$. By (iii), $A_{\beta_0}^n = A_{\beta_0+1}^n$, so by (b) for $h > n$ we have $A_\alpha^n = A_\beta^n = A_{\beta_0}^n = A_{\beta_0+1}^n \le A_{\beta_0+1}^h = A_\alpha^{h0}$.

This completes the proof of the lemma. ∎

Remark: From the proof it is clear that (by changing the well-ordering of the elements of A) we may assume that for any fixed α, given A_α, the subgroup $A_{\alpha+1}$ can be changed so as to contain any prescribed countable set of elements of A.

4. Separative chains

We utilize (3.1) in order to prove the main result (4.2) on filtrations of torsion-free groups. To begin with, we prove an auxiliary lemma.

LEMMA 4.1: *Assuming $V = L$, let G be a torsion-free group, and A and H_m ($m < \omega$) subgroups of G such that H_m and $A + H_m$ ($m < \omega$) are balanced in G ($H_0 = 0$). If $|A| = \kappa \geq \aleph_1$, then there is a continuous well-ordered ascending chain*

$$(3) \qquad 0 = A_0 < A_1 < \cdots < A_\alpha < \cdots \quad (\alpha < \kappa)$$

of pure subgroups of A such that

(a) $\bigcup_{\alpha<\kappa} A_\alpha = A$;
(b) $|A_\alpha| < \kappa$ *for all* $\alpha < \kappa$;
(c) *for each m and α, $A_\alpha + H_m$ is a B^∞-subgroup in G;*
(d) $A_{\alpha+1}/A_\alpha$ *is of* rank $\leq \aleph_1$ *for each* $\alpha < \kappa$.

Proof: If $\kappa = \aleph_1$, there is nothing to prove, so assume $\kappa > \aleph_1$. In the transfinite induction, we distinguish two cases according as $\kappa = \lambda^+$ with $\mathrm{cf}\lambda = \omega$ or not.

CASE 1: *κ is any cardinal, not of the form $\kappa = \lambda^+$ with $\mathrm{cf}\lambda = \omega$.* Manifestly, there is a chain (3) satisfying (a) and (b). GCH and the hypothesis on κ imply that if $|A_\alpha| < \kappa$, then also $|A_\alpha^{\aleph_0}| < \kappa$. Applying (2.3), a standard back-and-forth argument permits us to change this chain by dropping terms so as to preserve (a) and (b), and to make $A_\alpha + H_m$ ($m < \omega$) into balanced subgroups in G for $\mathrm{cf}\alpha \neq \omega$, and B^1-subgroups for $\mathrm{cf}\alpha = \omega$. We want to refine this chain to satisfy all the conditions by inserting subgroups between each adjacent A_α and $A_{\alpha+1}$.

If $\mathrm{cf}\alpha \neq \omega$, then apply the induction hypothesis to the balanced subgroup $A_{\alpha+1}$ (which has cardinality $\mu < \kappa$) in the role of A, and to the subgroups $A_\alpha + H_m$ ($m < \omega$) in the role of H_m ($m < \omega$). Thus we conclude that there exists a continuous well-ordered ascending chain of subgroups B_σ ($\sigma < \mu$) of G ($B_0 = 0$) with union $A_{\alpha+1}$ such that for all $\sigma < \mu$, $m < \omega$, $B_\sigma + A_\alpha + H_m$ are B^∞-subgroups of G, and $B_{\sigma+1}/B_\sigma$ is of rank $\leq \aleph_1$ for each $\sigma < \mu$. Set $B_{\alpha\sigma} = B_\sigma + A_\alpha$ for $\sigma < \mu$. These are evidently B^∞-subgroups of G such that $B_{\alpha\sigma} + H_m$ are all B^∞-subgroups of G. This leads us to a continuous well-ordered ascending chain of B^∞-subgroups

$$A_\alpha = B_{\alpha 0} \leq B_{\alpha 1} \leq \cdots \leq B_{\alpha\sigma} \leq \cdots < A_{\alpha+1}$$

where $B_{\alpha\sigma+1}/B_{\alpha\sigma}$ has rank at most the rank of $B_{\sigma+1}/B_\sigma$ which is $\leq \aleph_1$ for each $\sigma < \mu$.

If $\mathrm{cf}\alpha = \omega$, then write $A_\alpha = \bigcup_{n<\omega} A_{\alpha_n}$ with ordinals α_n such that $\sup \alpha_n = \alpha$ and $\mathrm{cf}\alpha_n \neq \omega$. Again, we apply the induction hypothesis to the balanced subgroup $A_{\alpha+1}$ of cardinality $\mu < \kappa$ (in place of A) and to the balanced subgroups $A_{\alpha_n} + H_m$ (in place of the H_m) to conclude that there exists a continuous well-ordered ascending chain of subgroups B_σ ($\sigma < \mu$) of G ($B_0 = 0$) with union $A_{\alpha+1}$ such that $B_\sigma + A_{\alpha_n} + H_m$ are all B^∞-subgroups of G and $B_{\sigma+1}/B_\sigma$ is of rank $\leq \aleph_1$ for each $\sigma < \mu$ and $n, m < \omega$. As before, we set $B_{\alpha\sigma} = B_\sigma + A_\alpha$ for $\sigma < \mu$ in order to obtain a chain between A_α and $A_{\alpha+1}$ as desired.

CASE 2: *$\kappa = \lambda^+$ where $\mathrm{cf}\lambda = \omega$.* Appeal to (3.1) with $A = G$ to conclude the existence of pure subgroups A_α, A_α^n ($\alpha < \lambda^+, n < \omega$), and in case $\mathrm{cf}\alpha = \omega$, pure subgroups A_α^{nk} ($k < \omega$) satisfying conditions (a)–(f) of (3.1). If (3) is the chain of these A_α, then conditions (a)–(c) above are evidently satisfied. To refine the chain between A_α and $A_{\alpha+1}$, we proceed as follows. For each $n < \omega$ apply the induction hypothesis to the balanced subgroup $A_{\alpha+1}^n$ of cardinality λ (playing the role of A) and to the countable set of balanced subgroups $A_\alpha^k + A_{\alpha+1}^{n-1} + H_m$ ($n, m < \omega$) or $A_\alpha^{lk} + A_{\alpha+1}^{n-1} + H_m$ ($n, k, m < \omega$) according as $\mathrm{cf}\alpha \neq \omega$ or $\mathrm{cf}\alpha = \omega$ (playing the roles of H_m). Note that condition (f) in (3.1) guarantees that all these subgroups are balanced in G. We obtain a continuous well-ordered ascending chain of B^∞-subgroups B_σ^n of G ($B_0^n = 0$) with union $A_{\alpha+1}^n$ such that the rank of $B_{\sigma+1}^n/B_\sigma^n$ is $\leq \aleph_1$. These countably many chains can be merged into a single chain by replacing B_σ^n by $B_\sigma^n + A_{\alpha+1}^{n-1}$ and letting the $n+1$st chain follow the nth chain. The combined chain clearly has $A_{\alpha+1}$ for its union such that the sum of every member of this chain with any of the indicated balanced subgroups is a B^∞-subgroup of G. This chain can now be handled just as in Case 1 to intercalate a chain of B^∞-subgroups $B_{\alpha\sigma}$ between A_α and $A_{\alpha+1}$. In this way, we are led to a chain between A_α and $A_{\alpha+1}$ with the desired properties. ∎

We are now in a position to verify the crucial result which enables us to pass the present barrier: the cardinal $\aleph_\omega$.

THEOREM 4.2: *Assume $V = L$. Let G be a torsion-free group and let κ denote its rank. There exists a continuous well-ordered ascending chain*

$$0 = G_0 < G_1 < \cdots < G_\nu < \cdots \tag{4}$$

of subgroups of G such that

(i) $\bigcup_{\nu<\kappa} G_\nu = G$;

(ii) *for each $\nu < \kappa$, G_ν has rank $< \kappa$;*

(iii) *for each $\nu < \kappa$, G_ν is separative in G;*

(iv) *for each $\nu < \kappa$, $G_{\nu+1}/G_\nu$ is torsion-free of rank 1.*

Proof: The claim is obvious whenever κ is finite or countably infinite. For $\kappa = \aleph_1$, this follows simply from (2.5). For $\kappa > \aleph_1$, we infer from the preceding lemma that there exists a chain like (4) where the groups are B^∞-subgroups of G and the factors are of ranks $\leq \aleph_1$. We can refine this chain by inserting between each consecutive term a chain of pure subgroups such that all the factors of this chain will be of rank 1. An appeal to (2.5) completes the proof. ∎

The last result leads us at once to a proof of Theorem II:

THEOREM 4.3 ($V = L$): $\mathrm{Bext}^2(G,T) = 0$ *holds for all torsion-free groups G and torsion groups T.*

Proof: If G admits a chain as stated in (4.2), then it satisfies the hypotheses of [1, Thm 6.3]. Hence the claim follows from the cited result. ∎

Obviously, the following corollary is equivalent to (4.3):

COROLLARY 4.4: *In L, balanced subgroups of B_1-groups, in particular, balanced subgroups of completely decomposable groups are B_1-groups.*

5. Axiom-3 families of separative subgroups

Let P be a property of subgroups. A group G is said to satisfy the **3rd Axiom of Countability** for subgroups of property P if there is a family $\mathcal{C}$ of subgroups of G with property P (called an **Axiom-3 family**) such that (i) $0, G \in \mathcal{C}$; (ii) if $\{H_i \mid i \in I\}$ is a subset of $\mathcal{C}$, then the subgroup generated by $\{H_i \mid i \in I\}$ belongs to $\mathcal{C}$; (iii) if $H \in \mathcal{C}$ and X is a countable subset of G, then there is a $K \in \mathcal{C}$ containing both H and X such that K/H is countable.

The idea of building an Axiom-3 family from a chain with countable factors, due to Hill [13], can be utilized for separative subgroups. In fact, from (4.2) a stronger assertion can be derived:

THEOREM 5.1: *In L, every torsion-free group admits an Axiom-3 family of separative subgroups.*

Proof: Let G be a torsion-free group of rank κ. By (4.2), G is the union of a continuous well-ordered chain (4) of separative subgroups of ranks $< \kappa$ with rank one quotients $G_{\nu+1}/G_\nu$. For each $\nu < \kappa$, and for each coset $g + G_\nu$ in $G_{\nu+1}$, consider a countable set $\{g + a_n |\ n < \omega\}$ with $a_n \in G_\nu$ such that $\{\chi(g + a_n)|\ n < \omega\}$ is cofinal in $\{\chi(g + a)|\ a \in G_\nu\}$. Let B_ν denote the subgroup of $G_{\nu+1}$ which is generated by the pure subgroups $\langle g + a_n\rangle_*(n < \omega)$ for each coset $g + G_\nu$ in $G_{\nu+1}$. Thus B_ν is a countable subgroup of $G_{\nu+1}$ satisfying $G_{\nu+1} = G_\nu + B_\nu$.

A subset S of κ will be called **closed** if for each μ in S we have

$$G_\mu \cap B_\mu \leq \langle B_\nu |\ \nu \in S, \nu < \mu\rangle.$$

Lemmas 5.5, 5.4 and 5.6 in [1] show respectively that

(1) the union of any number of closed subsets of κ is closed, and

(2) for a closed subset S of κ, the subgroup $G(S) = \langle B_\nu |\ \nu \in S\rangle$ is pure in G.

(3) Every countable subset F of κ is contained in a countable closed subset.

Let the family $\mathcal{C}$ consist of all subgroups of the form $G(S)$ with S closed in κ. This $\mathcal{C}$ will be a desired family if we can verify that $G(S)$ is separative in G provided that S is closed in κ.

Observe that every $g \neq 0$ in G defines an ordinal $\nu(g) < \kappa$ such that $G_{\nu(g)+1}$ is the first member of (4) containing g. Given $g \in G \setminus G(S)$, we have to find a countable cofinal subset in the set $\{\chi(g + x)|\ x \in G(S)\}$. We induct on $\nu(g)$.

To start the induction, let μ be the smallest index such that $\mu \notin S$; then $G_\mu \leq G(S)$. Let $\nu(g) = \mu$. There is a countable subset $\{\chi(g + a_n)|\ n < \omega\}$ cofinal in $\{\chi(g + x)|\ x \in G_\mu\}$ where evidently $a_n \in G(S)$. We claim that $\{\chi(g + a_n)|\ n < \omega\}$ is cofinal in $\{\chi(g + x)|\ x \in G(S)\}$. Given $x \in G(S)$, we have to find an $n < \omega$ such that $\chi(g + x) \leq \chi(g + a_n)$. We use induction on $\nu(x) = \lambda$. If $\lambda < \mu$, we are done by the choice of the a_n. Note that $\lambda = \mu$ is impossible. In fact, this would mean $x = x_0 + x_1$ with $x_0 \in \sum B_\nu$ $(\nu < \mu)$ and $0 \neq x_1 \in \sum B_\nu$ $(\nu > \mu)$, thus $x_1 = x - x_0 \in G_\mu$. If we write $x_1 = y_1 + \cdots + y_k$ with $\lambda_i = \nu(y_i) \in S$ and $\lambda_1 < \cdots < \lambda_k$ such that λ_k is minimal, then from $y_k \in G_{\lambda_k} \cap B_{\lambda_k} \leq \langle B_\nu |\ \nu \in S,\ \nu < \lambda_k\rangle$ we derive a contradiction. Finally, let $\lambda > \mu$; then necessarily $\lambda \in S$. Hence there is a $b_m \in G_\lambda$ such that $x + b_m \in B_\lambda$ and $\chi(g + x) \leq \chi(x + b_m)$. Then $\chi(g + x) \leq \chi(g - b_m)$. Here $b_m \in G(S)$, $\nu(b_m) < \lambda$, so by induction hypothesis $\chi(g - b_m) \leq \chi(g + a_n)$ for some n. We obtain the desired $\chi(g + x) \leq \chi(g + a_n)$.

In the second step of our induction, we can assume that $\nu(g)$ is minimal for the elements in the coset $g + G(S)$. Then $\nu(g) \notin S$, since for the coset $g + G_{\nu(g)}$ we have selected a countable set $\{g + a_n |\ n < \omega\}$ and $\nu(g) \in S$ would imply that a_n was a representative of the same coset with a smaller $\nu(a_n)$. By induction hypothesis, for each $n < \omega$, there is a countable cofinal set $\{\chi(a_n + a_{nk})|\ k < \omega\}$ in $\{\chi(a_n + x)|\ x \in G(S)\}$. We intend to show that the set $\{\chi(g - a_{nk})|\ n, k < \omega\}$ is cofinal in $\{\chi(g + x)|\ x \in G(S)\}$. As above, we induct on $\nu(x) = \lambda$. If $\lambda < \mu(g)$, then $\chi(g + x) \leq \chi(g + a_n)$ for some $n < \omega$; hence we have $\chi(g + x) \leq \chi(a_n - x)$. As $\chi(a_n - x) \leq \chi(a_n + a_{nk})$ for some k, we obtain $\chi(g + x) \leq \chi(g - a_{nk})$. The proof above applies to show that $\lambda = \mu(g)$ is impossible. The case $\lambda > \mu(g)$ can be handled as above to complete the proof that the set $\{\chi(g - a_{nk})|\ n, k < \omega\}$ is cofinal in $\{\chi(g + x)|\ x \in G(S)\}$. ∎

6. Separative subgroups and balanced extensions

Equipped with the necessary information about chains of separative subgroups in torsion-free groups, we can turn our attention to Butler groups of large cardinalities. We concentrate on the relationship between B_1- and B_2-groups. Actually, we could stop right here and delegate the balance of the proof of Theorem I to [8]. Instead, we show how some arguments in [8] can be simplified or improved.

Our discussion starts with a couple of preliminary lemmas.

LEMMA 6.1: *Let $0 \to A \to B \to C \to 0$ be an exact sequence of torsion-free groups, where A is separative in B and C is of rank 1. Then there is a balanced-exact sequence*

$$0 \to K \to A \oplus X \xrightarrow{\varphi} B \to 0$$

where X is completely decomposable of countable rank, and K is isomorphic to a pure subgroup of X.

Proof: For each coset $b + A$ ($b \in B$), pick a cofinal subset $\{\chi(b + a_n)|\ n < \omega\}$ (depending on b) in the set $\{\chi(b + a)|\ a \in A\}$, and let X_n be rank one groups with $x_n \in X_n$ of characteristic $\chi(b + a_n)$. Define X as the direct sum of all these countably many X_n, for all cosets and all n, and define φ via $\varphi(a) = a$ for $a \in A$, $\varphi(x_n) = b + a_n$. Then φ is surjective, since the characteristic of $b + A$ is the union of the characteristics $\chi(b + a_n)$ for $n < \omega$.

Let η: $J \to B$ be a homomorphism with J a subgroup of $\mathbb{Q}$. Assuming $1 \in J$, let $\eta 1 = b + a$ for some $b \in B \setminus A$, $a \in A$. Choose n such that $\chi(x_n) =$

$\chi(b+a_n) \geq \chi(b+a)$. Then $\chi(a-a_n) \geq \chi(b+a)$, so the correspondence $1 \mapsto x_n + a - a_n$ extends to a homomorphism $\xi\colon J \to A \oplus X$. This evidently satisfies $\varphi\xi = \eta$.

From the exact sequence $0 \to A \cap \varphi X \to A \oplus \varphi X \to B \to 0$ we derive that K is the inverse image of the pure subgroup $A \cap \varphi X$ of φX under φ. Hence K is as stated. ∎

The real significance of separative subgroups is that it renders possible the verification of the next lemma which is crucial in the proof of (6.3).

LEMMA 6.2: *Suppose* $0 \to T \to G \to A \to 0$ *is a balanced-exact sequence where* T *is a torsion group and* A *is a separative subgroup in a torsion-free group* B *with* $C = B/A$ *of rank 1. Then there exists a commutative diagram with balanced-exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T & \longrightarrow & H & \longrightarrow & B & \longrightarrow & 0. \end{array}$$

Proof: Choose a balanced-exact sequence $0 \to K \to A \oplus X \xrightarrow{\varphi} B \to 0$ as indicated in (6.1). This induces an exact sequence

$$\mathrm{Bext}^1(B,T) \to \mathrm{Bext}^1(A \oplus X, T) = \mathrm{Bext}^1(A,T) \to \mathrm{Bext}^1(K,T).$$

The last Bext vanishes in view of (6.1), since pure subgroups of countable completely decomposable groups are Butler groups (see e.g. [5]). Hence every balanced extension of T by $A \oplus X$, in particular, the extension $0 \to T \to G \oplus X \to A \oplus X \to 0$, is induced by a balanced extension of T by B. In other words, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & G \oplus X & \longrightarrow & A \oplus X & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T & \longrightarrow & H & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

with balanced-exact bottom row. Dropping the X's, we get a diagram as desired. ∎

We can now apply the results to Butler groups.

THEOREM 6.3: *Assume $V = L$. A B_1-group G of any cardinality κ admits a continuous well-ordered ascending chain*

$$0 = G_0 < G_1 < \cdots < G_\nu < \cdots \quad (\nu < \kappa) \tag{5}$$

of pure subgroups such that

(i) $\bigcup_{\nu<\kappa} G_\nu = G$;
(ii) *each G_ν is separative in G;*
(iii) *each G_ν has cardinality $< \kappa$;*
(iv) *$G_{\nu+1}/G_n$ is of rank 1 for each $\nu < \tau$;*
(v) *for every $\nu < \kappa$, G_ν is a B_1-group.*

Proof: In view of (4.2), all what we have to verify is that the groups in the chain (4) are B_1-groups whenever G is a B_1-group. Because of (6.2), a balanced-exact sequence $0 \to T \to H_\nu \to G_\nu \to 0$ with T torsion can be embedded in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & H_\nu & \longrightarrow & G_\nu & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T & \longrightarrow & H_{\nu+1} & \longrightarrow & G_{\nu+1} & \longrightarrow & 0 \end{array} \tag{6}$$

with balanced-exact bottom row. Since the images of the mappings $G_\nu \to G_{\nu+1}$ are pure subgroups, the direct limit $0 \to T \to H \to G \to 0$ of the balanced-exact sequences $0 \to T \to H_\nu \to G_\nu \to 0$ is easily seen to be again balanced-exact. In this way, we obtain a commutative diagram like (6) with the bottom sequence replaced by the direct limit. If G is a B_1-group, then the direct limit splits, and therefore so does the top sequence. Hence G_ν is likewise a B_1-group. ∎

Note that the proof shows that if G_ν is a B_1-group, then any chain (4) satisfying (i)–(iv) also satisfies (v).

7. B_3-Groups

A subgroup A of a group G with G/A torsion-free is called **prebalanced** if the following condition is satisfied: for every rank one (pure) subgroup C/A of G/A, there exists a Butler subgroup B of finite rank in C such that $C = A + B$. (For homological properties of prebalancedness, we refer to Fuchs–Metelli [12].) The subgroup A is **decent** in G if the same holds for all finite rank pure subgroups C/A of G/A; see Albrecht–Hill [1].

We shall need the equivalence of B_2-groups and B_3-groups, as asserted in Albrecht–Hill [1]. Since the proof of their Lemma 5.7 was based on the incorrect claim that every countable set of ordinals can be arranged in an ascending chain of type ω ($\omega + \omega$ is a counterexample), we cannot rely on the proof in [1]. Here we shall prove a somewhat more general statement.

Define a subgroup H of G to have **property** P if it is a pure subgroup of G, and for each pure subgroup K of G which contains H as a finite corank subgroup, $K = H + B$ holds for some finite rank subgroup B of G.

THEOREM 7.1: *If G is the union of a continuous well-ordered ascending chain of pure subgroups, $0 = H_0 < H_1 < \cdots < H_\nu < \cdots$ ($\nu < \mu$), such that for each $\nu + 1 < \mu$, $H_{\nu+1} = H_\nu + B_\nu$ with a finite rank subgroup B_ν, then G satisfies the 3rd axiom of countability for subgroups of property P.*

Proof: A subset S of μ will be called **closed** if for each λ in S we have $H_\lambda \cap B_\lambda \leq \langle B_\nu \mid \nu \in S, \nu < \lambda\rangle$. We have (1) and (2) as in the proof of (5.1), but (3) should be replaced by the stronger claim:

(3′) Every finite subset F of μ is contained in a finite closed subset.

We prove this by induction on the maximal member of the finite set F. If this maximum is 0, then $F = \{0\}$ is trivially closed. Let $\lambda > 0$ be the largest ordinal in F, and assume the claim true for finite subsets $\subset \lambda$. In view of (1), it is enough to show that $\{\lambda\}$ is contained in a finite closed subset. As the subgroup $H_\lambda \cap B_\lambda$ is of finite rank, it contains a finite maximal family of independent elements $x_1, ..., x_k$. The x_i's are all in H_λ, so we can find a finite subset S' of λ, such that x_i is in $G(S')$ for $i \leq k$. By induction hypothesis we can assume S' closed. (There is no loss of generality in assuming that this finite set is still included in λ, because it follows easily from the definition of closed sets that $S \cap \lambda$ is closed for every λ whenever S is closed.) We claim that $S = S' \cup \{\lambda\}$ is closed, and hence it is the required finite closed set. As S' is closed, it is enough to check the definition only for λ. By (2), $G(S')$ is a pure subgroup of G, hence it contains $H_\lambda \cap B_\lambda$. We are done with the proof of claim (3′).

(4) For any closed subset S of μ, the subgroup $G(S)$ has property P.

Let K be a pure subgroup of G that contains $G(S)$ such that $K/G(S)$ is of finite rank. Then there is a finite subset $\{x_1, ..., x_k\}$ in K such that $\langle G(S), x_1, ..., x_k\rangle$ is an essential subgroup of K. By (3′) we can find a finite closed subset T of μ satisfying $\{x_1, ..., x_k\} \subset G(T)$. Then as a pure subgroup,

$G(S \cup T) = G(S) + G(T)$ contains K, and therefore $K = G(S) + (K \cap G(T))$ holds where the second subgroup is evidently of finite rank.

Define the family $\mathcal{C}$ to consist of all subgroups of G that are of the form $G(S)$ with S a closed subset of μ. Then the members of $\mathcal{C}$ have property P, while properties (i)–(iii) of families with 3rd axiom of countability are readily checked. ∎

Applying (7.1) to the case in which the finite rank subgroups B_ν in the definition of property P are Butler groups (i.e. property P means being decent in G), we are led to the desired conclusion:

THEOREM 7.2: *A torsion-free group is a B_2-group if and only if it is a B_3-group.* ∎

From the definition it is straightforward to verify that B_3-groups are finitely Butler in the sense that all of their pure subgroups of finite rank are Butler. Hence B_2-groups are finitely Butler.

Before stating the following corollary, we remind the reader of a definition. For an infinite cardinal κ, by a $G(\kappa)$-**family** in the group G is meant a collection $\mathcal{C}$ of subgroups of G such that (i) $0, G \in \mathcal{C}$; (ii) $\mathcal{C}$ is closed under unions of chains; (iii) if $A \in \mathcal{C}$ and X is any subset of G of cardinality $\leq \kappa$, then there is a $B \in \mathcal{C}$ that contains both A and X, and satisfies $|B/A| \leq \kappa$. Manifestly, every Axiom-3 family is a $G(\aleph_0)$-family.

COROLLARY 7.3 (CH): *Every B_2-group admits a $G(\aleph_1)$-family of balanced subgroups.*

Proof: Let G be a B_2-group, and $\mathcal{C}$ a $G(\aleph_0)$-family of decent subgroups. Let $\mathcal{B}$ denote the subset $\{B \in \mathcal{C} \mid B$ is balanced in $G\}$.

To verify (ii) for $\mathcal{B}$, in view of (2.2) it suffices to show that the union B of a countable ascending chain of balanced subgroups B_n $(n < \omega)$ is again balanced in G whenever B is prebalanced in G. Let E be a pure subgroup of G such that $B < E \leq G$ and E/B is of rank 1. On account of prebalancedness, $E = B + \sum E_i$ holds for subgroups E_i of rank 1 $(i = 1, \dots, k)$. Write $E_i = \langle e + b_i \rangle_*$ for some $e \in E$ and $b_i \in B$; then $b_1, \dots, b_k \in B_m$ for some index m. Clearly, B_m is balanced of corank 1 in $B' = B_m + \sum E_i$, thus $B' = B_m \oplus X$ for a rank 1 group $X = \langle e + b \rangle_*$ for a suitable $b \in B_m$. But then $E = B \oplus X$ establishes the balancedness of B in G.

To check (iii), let $B \in \mathcal{B}$ be a balanced subgroup and X a subgroup of cardinality $\leq \aleph_1$. By (2.3), there is a balanced subgroup C/B of G/B of cardinality $\leq \aleph_1$ containing $(X + B)/B$. A routine back-and-forth argument convinces us that C can be chosen so as to belong to $\mathcal{C}$. C is balanced in G, so $C \in \mathcal{B}$. ∎

8. Regular cardinals

We need some more preparatory results to prove that B_1-groups of arbitrary cardinality are B_2-groups. First, we deal with the case of regular cardinals. The key lemma is a version of a lemma by Eklof–Fuchs [10], due to Dugas–Hill–Rangaswamy [8]. Recall that a pure subgroup A of a torsion-free group G is said to be a TEP-subgroup, or to have TEP in G (Torsion Extension Property), if every homomorphism $A \to T$ (T any torsion group) extends to a homomorphism $G \to T$.

LEMMA 8.1: *Let κ be an uncountable regular cardinal, and $0 = A_0 < A_1 < \cdots < A_\nu < \cdots$ ($\nu < \kappa$) a continuous well-ordered ascending chain of pure subgroups of a torsion-free group A such that*

(a) $\bigcup_{\nu<\kappa} A_\nu = A$;

(b) $|A_\nu| < \kappa$ *for all* $\nu < \kappa$;

(c) *for each* $\nu < \kappa$, A_ν *is a* B_1*-group.*

If A is a B_1-group, then the set

$$E = \{\nu < \kappa \mid \exists \mu > \nu \text{ such that } A_\nu \text{ is not a TEP-subgroup in } A_\mu\}$$

is not stationary in κ. ∎

We require a generalization of a crucial lemma by Dugas–Rangaswamy [7]. The proof of (2.2) in Fuchs–Metelli [12] furnishes us with what we need here:

LEMMA 8.2: *Let G be a B_1-group and A a separative subgroup of finite corank in G. A has* TEP *in G if and only if $G = A + B$ for a finite rank Butler group B, i.e. A is decent in G.*

We can now prove:

THEOREM 8.3 ($V = L$): *Suppose the cardinality κ of the torsion-free group G is an uncountable regular cardinal. If G is a B_1-group, then it admits a continuous well-ordered ascending chain of pure subgroups, $0 = A_0 < A_1 < \cdots < A_\alpha < \cdots$ ($\alpha < \kappa$), such that*

(i) $\bigcup_{\alpha<\kappa} A_\alpha = G$;
(ii) $|A_\alpha| < \kappa$ *for all* $\alpha < \kappa$;
(iii) *each* A_α *is a prebalanced subgroup in* G;
(iv) *both* A_α *and* $A_{\alpha+1}/A_\alpha$ *are* B_1*-groups for each* $\alpha < \kappa$.

Proof: In view of (6.3), G has a continuous well-ordered ascending chain of separative, B_1-subgroups A_α satisfying (i) and (ii). Moreover, (8.1) guarantees that, by dropping to a suitable cub of indices in κ if necessary, we can assume that the subgroups A_α in the chain are TEP-subgroups of G.

In order to show that in the arising chain, each A_α is prebalanced in A, choose any pure subgroup A'_α of G that contains A_α with A'_α/A_α of rank 1. Without loss of generality A_α may be assumed to be one of the subgroups G_μ in the chain (5). This chain can be modified by replacing $G_{\mu+1}$ by A'_α and every $G_{\lambda+1}$ ($\lambda > \mu$) by the purification of $G_\lambda + A'_\alpha$. These subgroups are again separative in G, thus (6.3) (in particular, the remark following (6.3)) ensures that A'_α is a B_1-group. (8.2) implies the prebalancedness of A_α in A'_α, and hence — since A'_α was arbitrary — in G. This establishes (iii).

Applying Proposition 2.1 in Fuchs–Metelli [12] to the prebalanced-exact sequence $0 \to A_\alpha \to A_{\alpha+1} \to A_{\alpha+1}/A_\alpha \to 0$, from the TEP-property of A_α in $A_{\alpha+1}$ we derive that $A_{\alpha+1}/A_\alpha$ is a B_1-group. Hence (iv) holds. ∎

9. Singular cardinals

We now turn our attention to the singular cardinal case. Shelah's Singular Compactness Theorem will be applied in the form phrased by Hodges [14]. This yields a more direct approach than the method employed by Dugas–Hill–Rangaswamy [8] in that it applies to the group itself rather than to a balanced-projective resolution of the group.

THEOREM 9.1: *Let* G *be a torsion-free group whose cardinality is a singular cardinal* λ. *If every balanced subgroup of* G *whose cardinality is* $< \lambda$ *is a* B_3*-group, then* G *itself is a* B_3*-group.*

Proof: Hodges' theorem [14, Thm 5] states that if his Axioms I–V (see below) hold for a set G and a cardinal $\lambda' < \lambda = |G|$, λ a singular cardinal, and if for every cardinal κ in some sets of cardinals with supremum λ, Player I has no winning strategy in the κ^+-Kueker game on G, then G is "free".

Thus we are given an arbitrary set G (whose cardinality λ is a singular cardinal) along with a family $S(G)$ of subsets of G; these subsets are called subalgebras of G. In our case, G will be a torsion-free group of cardinality λ, and $S(G)$ the collection of all subgroups of G. We also have to choose an infinite cardinal $\lambda' < \lambda$; our choice is $\lambda' = \aleph_0$.

Moreover, we need a distinguished collection of subalgebras B called "free subalgebras" such that each of them has at least one basis, i.e. a certain collection of subalgebras in B. In our case, a subgroup B of G is "free" if it is a B_3-group and a "basis" for B will be an Axiom-3 family of decent subgroups, say, $\mathcal{F} = \{H_i \mid i \in I\}$.

Let us check if all the axioms in Hodges' paper [14] are satisfied.

AXIOM I: *$S(G)$ is fully closed unbounded.*

This means that $S(G)$ is closed under unions of ascending chains and that every subset B of G is contained in a member of $S(G)$ whose cardinality is $\leq |B| + \aleph_0$. All of this is obvious in our case since $S(G)$ is the collection of all subgroups of G.

AXIOM II: *If $\mathcal{F}$ is a basis of B, then $\mathcal{F}$ is a closed unbounded collection of subalgebras of G included in B, fully closed unbounded in B.*

This follows from the fact that $\mathcal{F}$ is a witness to B being a B_3-group.

AXIOM III: *If $\mathcal{F}$ is a basis for B, and if $C \in \mathcal{F}$, then the set $\{D \mid D \in \mathcal{F}, D \leq C\}$ is a basis for C.*

This is clear, since subgroups of C decent in B are decent in C.

AXIOM IV: *If C is a member of some basis $\mathcal{F}$ for B, and if $\mathcal{F}'$ is a basis for C, then there is a basis for B, whose restriction to C is $\mathcal{F}'$.*

If in $\mathcal{F}$ we replace the set $\{D \mid D \in \mathcal{F}, D$ subset of $C\}$ by $\mathcal{F}'$, then we get a required basis of B, since decency is a transitive property.

AXIOM V: *Suppose we are given a continuous well-ordered ascending chain of subalgebras of G, $\{B_\nu \mid \nu < \kappa\}$, along with a chain of bases, $\{\mathcal{F}_\nu \mid \nu < \kappa\}$, such that $\mathcal{F}_\nu$ is a basis for B_ν and if $\nu < \mu < \kappa$, then $\mathcal{F}_\mu$ restricted to B_ν is $\mathcal{F}_\nu$ (in particular B_ν is in $\mathcal{F}_\mu$). Then the union $B = \bigcup\{B_\nu \mid \nu < \kappa\}$ has a basis consisting of unions of chains of the form $\bigcup\{H_\nu \mid \nu < \kappa\}$ where $H_\nu \in \mathcal{F}_\nu$ for $\nu < \kappa$.*

Define $\mathcal{F} = \bigcup\{\mathcal{F}_\nu | \nu < \kappa\}$. All what we have to verify is that, for $\nu < \kappa$, each $H_\nu \in \mathcal{F}_\nu$ is decent in B. Let K be any pure subgroup of B containing H_ν such that K/H_ν is of finite rank; then there is a $\mu < \kappa$ $(\nu < \mu)$ such that $K \leq B_\mu$. H_ν being decent in B_μ, we have $K = H_\nu + K'$ for a finite rank Butler subgroup K', proving that H_ν is decent in B as well.

Let $\kappa < \lambda$ be a regular cardinal. Recall that by a κ-Kueker game on G is meant the following game between two players. Players I and II choose alternately subgroups B_α of G (to form a continuous well-ordered ascending chain) such that (1) each B_α is of cardinality $< \kappa$, (2) for each α, $B_{\alpha+1}$ contains B_α, and (3) at limit ordinals α Player I ought to choose $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$. Player II wins if and only if the subgroup $B_\kappa = \bigcup_{\alpha<\kappa} B_\alpha$ is free.

In order to apply Hodges' theorem we need to show that there are arbitrarily large cardinals $\kappa < \lambda$ such that "Player I has no winning strategy" in the κ^+-Kueker game on G. We can even show that "Player II has a winning strategy".

Assume that κ is any uncountable cardinal $< \lambda$. In order to win in the κ^+-Kueker game on G, at his turn Player II plays by picking a balanced subgroup of G of cardinality κ that contains the previous subgroup played by Player I. Since κ^+ is uncountable, the final union is balanced in G, and has cardinality $< \lambda$. By assumption, it is a B_3-group. In other words, Player II is guaranteed a victory. This completes the proof that G is likewise a B_3-group. ∎

10. Proof of Theorem I

We are now in the possession of all the ingredients needed to finish the proof of our main result:

THEOREM 10.1: *Assuming $V = L$, B_1-groups of any cardinality are B_2-groups.*

Proof: By induction on the cardinality κ of the B_1-group. For $\kappa = \aleph_0$, this has been proved by Bican–Salce [5]; for a short proof see Fuchs–Metelli [12]. Let $\kappa > \aleph_0$ and assume the claim has been verified for torsion-free groups of smaller cardinalities.

CASE 1: *κ is a regular cardinal.* A B_1-group G of cardinality κ has a chain $0 = A_0 < A_1 < \cdots < A_\alpha < \cdots (\alpha < \kappa)$ with properties (i)–(iv) listed in (8.3). By induction hypothesis, each $A_{\alpha+1}/A_\alpha$ is a B_2-group, hence it has a continuous well-ordered ascending chain of prebalanced subgroups with rank 1 factors. The prebalanced subgroups C/A_α of $A_{\alpha+1}/A_\alpha$ lift to prebalanced subgroups C of G

(cf. Fuchs–Metelli [12]) to form a chain of prebalanced subgroups between A_α and $A_{\alpha+1}$. All the arising chains put together will yield a chain of prebalanced subgroups with rank 1 factors for the group G. Consequently, G is a B_2-group.

CASE 2: *κ is a singular cardinal.* Let G be a B_1-group of cardinality κ. Because of (4.5), all the balanced subgroups of G are B_1-groups. By induction hypothesis, the balanced subgroups of cardinalities $< \kappa$ are B_2-groups, and thus by (7.2) they are B_3-groups. A simple appeal to (9.1) completes the proof that G is a B_3-group. ∎

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